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TC OPERATIONS AND LATIN BRICKS*

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The *term condition* for an operation generalizes a cancellation property of linear functions on a module. It has had several applications in Universal Algebra. Operations satisfying the term condition are called *TC operations*. For a set A, we denote by $\mathrm{TC}_n(A)$ the set of all n-ary TC operations on A. Berman and McKenzie proved that for $|A|=2$ and $|A|=3$, the cardinality of $\mathrm{TC}_n(A)$ is, respectively, 2^{n+1} and $3^{2n+1}+3\cdot 2^{2n+1}-18n-6$, and that for $|A|=k\geq 4$, we have

$$k^{(k-1)n+1}\leq|\mathrm{TC}_n(A)|\leq 2k^{(k-1)n+1+m},$$

where $m=k^2+\cdots+k^{k-2}$.

In this paper we introduce the notion of a *Latin brick*, and show that counting TC operations on a certain set A is equivalent to counting TC Latin bricks defined on some sets that depend on A. In particular, this enables us to compute $|\mathrm{TC}_2(A)|$ for $|A|=4$. We also characterize TC Latin bricks defined on certain types of product sets; those Latin bricks are induced by Abelian groups.

1. The term condition

In the sequel, A is a nonempty set. For $n=1,2,\ldots$, a function from A^n into A is called an *n-ary operation* on A. The set $\{0,1,\ldots,n-1\}$ is denoted by $\underline{n}$.

Definition 1.1. Let $A_0,A_1,\ldots,A_n$ be nonempty sets, and $f:\prod_{i\in\underline{n}}A_i\to A_n$ be a function. For each nonvoid subset I of $\underline{n}$ and each $\boldsymbol{a}\in\prod_{j\in\underline{n}-I}A_j$, we define a function

$$\begin{aligned} f[\boldsymbol{a},I]:\textstyle\prod_{i\in I}A_i&\to A_n\\ \boldsymbol{x}&\mapsto f(\boldsymbol{a}\cup\boldsymbol{x}),\end{aligned}$$

where $\boldsymbol{a}\cup\boldsymbol{x}=(y_0,\ldots,y_{n-1})$ is defined by $y_i=a_i$ if $i\in\underline{n}-I$, and $y_i=x_i$ otherwise.

We call $f[\boldsymbol{a},I]$ an *I-derivative* of f; in particular, when I is a singleton $\{i\}$, we write $f[\boldsymbol{a},i]$ and call it an i-derivative.

Definition 1.2. A function $f:\prod_{i\in\underline{n}}A_i\to A_n$ is said to satisfy the *term condition*, or to be *TC*, if for every $i\in\underline{n}$, every two i-derivatives $f[\boldsymbol{a},i]$ and $f[\boldsymbol{b},i]$ are either equal, or different at each point (i.e. $f[\boldsymbol{a},i](x)\neq f[\boldsymbol{b},i](x)\ \forall x\in A_i$).

Following [3], we extend this definition to I-derivatives: for $m=1,2,\ldots$, f is

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TC^m if for each subset I of $\underline{n}$ of cardinality m, every two I-derivatives are either equal, or different at each point.

In particular, if $A_0 = \cdots = A_{n-1} = A_n$, f is called a TC, or TC^m operation. The set of n-ary TC operations on A is denoted by $TC_n(A)$.

Remarks 1.3. (1) f is obviously TC^m if $m \geq n$. In particular every unary operation is TC.

(2) TC^1 is the same as TC.

(3) It is not hard to see that TC^m implies TC^{m+1}.

(4) If f is TC^m, then any restriction f' of f to a subset $\coprod_{i \in n} B_i$, where $B_i \subset A_i \; \forall i$, is also TC^m (since the I-derivatives of f' are restrictions of I-derivatives of f).

Examples 1.4. (1) The operation q of a quasigroup (in particular of a group) is a binary TC operation for which two i-derivatives $q[a,i]$ and $q[b,i]$ are never equal unless $a = b$.

(2) The operation

$$\begin{aligned} f: \underline{5}^2 &\to \underline{5} \\ (x,y) &\mapsto x + y (\mathrm{mod}\ 3) \end{aligned}$$

is TC and satisfies $f[a,i] = f[b,i]$ iff $a \equiv b (\mathrm{mod}\ 3)$. (Note that 3 and 5 could be replaced by any pair of natural numbers.)

(3) Let $G_0, \ldots, G_{n-1}, G_n$ be groups. Then any group homomorphism $\varphi: \coprod_{i \in n} G_i \to G_n$ is TC.

Lemma 1.5. *Let $A_0, \ldots, A_n$, $B_0, \ldots, B_n$ be sets, $p_i : A_i \to B_i$ $(i = 0, \ldots, n-1)$ functions and $p_n : B_n \to A_n$ an injection. If $g: \coprod_{i \in n} B_i \to B_n$ is TC, then so is*

$$\begin{aligned} f: \textstyle\coprod_{i \in n} A_i &\to A_n \\ (a_0, \ldots, a_{n-1}) &\mapsto p_n g(p_0(a_0), \ldots, p_{n-1}(a_{n-1})). \end{aligned}$$

If, moreover, $p_0, \ldots, p_n$ are bijective, then g is TC if and only if f is TC.

Proof. Let $\boldsymbol{a} = (a_j)_{j \neq i}$, $\boldsymbol{b} = (b_j)_{j \neq i} \in \coprod_{j \neq i} A_j$ and $x \in A_i$, for some $i \in n$. Suppose that $f[\boldsymbol{a}, i](x) = f[\boldsymbol{b}, i](x)$. Then we have

$$\begin{aligned} & f(\boldsymbol{a} \cup x) = f(\boldsymbol{b} \cup x) \\ \Leftrightarrow \quad & p_n g((p_j(a_j))_{j \neq i} \cup p_i(x)) = p_n g((p_j(b_j))_{j \neq i} \cup p_i(x)) \\ \Leftrightarrow \quad & g((p_j(a_j))_{j \neq i} \cup p_i(x)) = g((p_j(b_j))_{j \neq i} \cup p_i(x)) \quad \text{because } p_n \text{ is injective} \\ \Leftrightarrow \quad & g[(p_j(a_j))_{j \neq i}, i] = g[(p_j(b_j))_{j \neq i}, i] \quad \text{by TC} \end{aligned}$$

which implies that $f[\boldsymbol{a}, i] = f[\boldsymbol{b}, i]$.

Now assume that $p_0, \ldots, p_n$ are bijections and that f is TC. Then by the first part of the lemma, the function

$$h : \prod_{i\in \underline{n}} B_i \to B_n$$
$$(b_0, \dots, b_{n-1}) \mapsto p_n^{-1} f(p_0^{-1}(b_0), \dots, p_{n-1}^{-1}(b_{n-1}))$$

is TC. But h is precisely g, hence g is TC. □

2. Latin bricks

The binary operation of a quasigroup has the property that all its i-derived functions are injective. We define a larger class of functions having this property.

Definition 2.1. We call a function $\beta : \prod_{i\in \underline{n}} A_i \to A_n$ a *Latin brick* if all its i-derived functions are injective.

Example 2.2. The function $f : \underline{2}^3 \to \underline{3}$, given by its $\underline{2}$-derived functions below, is a Latin brick. However, the relations $f(0,0,0) = f(1,1,0)$ and $f(0,0,1) \neq f(1,1,1)$ show that f is not TC.

$f[0,\underline{2}]$	0	1
0	0	1
1	2	0

$f[1,\underline{2}]$	0	1
0	1	0
1	0	2

Remark 2.3. As for TC functions, restrictions of Latin bricks on product subsets are still Latin bricks.

From now on, all Latin bricks considered will be TC Latin bricks.

Lemma 2.4. *A function $f : \prod_{i\in \underline{n}} A_i \to A_n$ is TC if and only if there exists a TC Latin brick*

$$\beta : \prod_{i\in \underline{n}} B_i \to A_n$$

such that for all $a_i \in A_i$ $(i = 0, \dots, n-1)$, we have

$$f(a_0, \dots, a_{n-1}) = \beta(p_0(a_0), \dots, p_{n-1}(a_{n-1})),$$

where $B_i \subset A_i$ and $p_i : A_i \to B_i$ are such that $p_i|_{B_i} = \mathrm{id}_{B_i}$ for $i = 0, \dots, n-1$.

Proof. ($\Leftarrow$) It suffices to apply Lemma 1.5, taking p_n to be the identity.

($\Rightarrow$) We define relations ϱ_i on A_i $(i = 0, \dots, n-1)$ by setting $(a_i, a_i') \in \varrho_i$ if there exists $\boldsymbol{a} \in \prod_{j\in \underline{n}-\{i\}} A_j$ such that $f(a_i \cup \boldsymbol{a}) = f(a_i' \cup \boldsymbol{a})$. The ϱ_i are clearly symmetric and reflexive; to see that they are also transitive, we prove the equivalence

$$(a_i, a_i') \in \varrho_i \quad \Leftrightarrow \quad \forall \boldsymbol{a} \in \prod_{j\in \underline{n}-\{i\}} A_j \ f(a_i \cup \boldsymbol{a}) = f(a_i' \cup \boldsymbol{a}). \qquad (1)$$

Indeed, if $f(a_i \cup c) = f(a_i' \cup c)$ for some $c \in \prod_{j \in \underline{n} - \{i\}} A_j$, then by TC^{n-1}, we have

$$f[a_i, \underline{n} - \{i\}] = f[a_i', \underline{n} - \{i\}],$$

which proves (1). Now for each $i \in \underline{n}$, choose one representative for every block of ϱ_i; call B_i the set of these representatives, and let $p_i : A_i \to B_i$, which sends x to the representative of the block to which x belongs. Define β to be the restriction of f to $\prod_{i \in \underline{n}} B_i$. By Remark 1.3(4), β is TC. Also, the restriction of p_i to B_i is clearly the identity. To see that β is a Latin brick, let $\beta[\boldsymbol{b}, i]$ be an i-derived function of β. We have, for all $x, y \in B_i$, $\beta[\boldsymbol{b}, i](x) = \beta[\boldsymbol{b}, i](y) \Leftrightarrow f[\boldsymbol{b}, i](x) = f[\boldsymbol{b}, i](y) \Leftrightarrow (x, y) \in \varrho_i \Leftrightarrow x = y$. Hence the i-derived functions of β are injective. Finally take $a_i \in A_i$ $(i = 0, \ldots, n-1)$. Since $(a_0, p_0(a_0)) \in \varrho_0$, we have

$$f(a_0, \ldots, a_{n-1}) = f(p_0(a_0), a_1, \ldots, a_{n-1}).$$

By the same tool, we obtain

$$\begin{aligned} f(p_0(a_0), a_1, \ldots, a_{n-1}) &= f(p_0(a_0), p_1(a_1), \ldots, a_{n-1}) \\ &\;\;\vdots \\ &= f(p_0(a_0), \ldots, p_{n-1}(a_{n-1})) \\ &= \beta(p_0(a_0), \ldots, p_{n-1}(a_{n-1})). \end{aligned}$$ ⌟

Note that the Latin brick of the previous lemma is not uniquely defined by f, since it depends on a certain choice of representatives of the ϱ_i-blocks. For a finite set $\underline{k}$, we will avoid this problem in the following way.

Definition 2.5. Let $f \in \mathrm{TC}_n(\underline{k})$. In the proof of Lemma 2.4, if we choose the representative of a ϱ_i-block to be the least element of that block, we obtain a unique TC Latin brick determined by f. We shall call it the TC *Latin brick associated to* f and will denote it by β_f.

Lemma 2.6. *Let $\gamma : \prod_{i \in \underline{n}} B_i \to k$, where $B_i \subset \underline{k}$ for $i = 0, \ldots, n-1$, be a TC Latin brick. Then there exists $f \in \mathrm{TC}_n(\underline{k})$ such that $\gamma = \beta_f$, if and only if $0 \in B_i$ for all $i \in \underline{n}$.*

Proof. ($\Rightarrow$) If $\gamma = \beta_f$ for some $f \in \mathrm{TC}_n(\underline{k})$, then for all $i = 0, \ldots, n-1$, 0 is in a certain ϱ_i-block and hence is its least element.

($\Leftarrow$) Suppose $0 \in B_i$ for all i. We define equivalences ϱ_i $(i = 0, \ldots, n-1)$ on $\underline{k}$ by their blocks $C_{i0}, \ldots, C_{im_i}$, where

$$B_i = \{x_{i0}, \ldots, x_{im_i}\} \quad \text{with } 0 = x_{i0} < \cdots < x_{im_i},$$

$$C_{ij} = \{x \in \underline{k} : x_{ij} \le x < x_{i,j+1}\} \quad \text{for } j = 0, \ldots, m_i - 1,$$

and

$$C_{im_i} = \{x \in \underline{k} : x_{im_i} \le x\}.$$

Next, define

$$p_i : \underline{k} \to B_i$$

$$x \mapsto \begin{cases} x_{ij} & \text{if } x < x_{im_i} \text{ and } x_{ij} \le x \le x_{i,j-1} \\ x_{im_i} & \text{if } x_{im_i} \le x \end{cases}$$

and

$$f : \underline{k}^n \to \underline{k}$$
$$(a_0, \dots, a_{n-1}) \mapsto \gamma(p_0(a_0), \dots, p_{n-1}(a_{n-1})).$$

By Lemma 2.4, f is TC, and it is easy to see that $\beta_f = \gamma$. □

Lemma 2.7. *Consider the equivalence $\beta = \{(f,g) \in \mathrm{TC}_n(\underline{k}) : \beta_f = \beta_g\}$ on $\mathrm{TC}_n(\underline{k})$. Then for each β-block $[f]$, where β_f is defined on $\prod_{i \in \underline{n}} B_i$, there is a bijection φ between $[f]$ and the set $\mathscr{R}$ of sequences $(\varrho_0, \dots, \varrho_{n-1})$ of equivalences on $\underline{k}$ such that $B_i = \{\min[x]_{\varrho_i} : x \in \underline{k}\}$ $(i = 0, \dots, n-1)$.*

Proof. We define a function $\varphi : [f] \to \mathscr{R}$, which assigns to $g \in [f]$ the sequence of equivalences of Lemma 2.4. Since $\beta_g = \beta_f$, φ is well defined. Now φ is surjective because for any $(\varrho_0, \dots, \varrho_{n-1}) \in \mathscr{R}$, we are able to construct a TC operation f such that $(\varrho_0, \dots, \varrho_{n-1})$ is the sequence of equivalences of f (see proof of Lemma 2.6). Finally, the injectivity of φ comes from the fact that, given β_g and the sequence of equivalences of g, we can reconstruct g. □

Lemma 2.8. *Let $B_0, \dots, B_{n-1}$, $C_0, \dots, C_{n-1}$ and A be nonempty sets, π a permutation of $\underline{n}$, and $\sigma_i : C_{\pi(i)} \to B_i$ $(i = 0, \dots, n-1)$ be bijections. Call $\mathscr{B}$ the set of all TC Latin bricks from $\prod_{i \in n} B_i$ into A, and $\mathscr{C}$ the set of TC Latin bricks from $\prod_{i \in \underline{n}} C_i$ into A. Then the function $\phi : \mathscr{B} \to \mathscr{C}$ such that*

$$\phi(\beta)(c_0, \dots, c_{n-1}) = \beta(\sigma_0(c_{\pi(0)}), \dots, \sigma_{n-1}(c_{\pi(n-1)}))$$

is a bijection.

Proof. We first prove that ϕ is well defined, i.e. that $\phi(\beta)$ is a TC Latin brick for all $\beta \in \mathscr{B}$. The i-derived functions of $\phi(\beta)$ are of the form $\phi(\beta)[\boldsymbol{c}, i]$ with

$$\phi(\beta)[\boldsymbol{c}, i](x) = \beta(\boldsymbol{b} \cup \sigma_{\pi^{-1}(i)}(x)), \quad \text{where } \boldsymbol{b} = (\sigma_j(c_{\pi(j)}))_{j \neq i}.$$

They are injective since the i-derived functions of β, as well as the σ_i's are injective. Now, $\phi(\beta)$ is TC; take $x, y \in C_i$. We have

$$\begin{aligned}
&\phi(\beta)[\boldsymbol{c}, i](x) = \phi(\beta)[\boldsymbol{c}', i](x) \\
&\Leftrightarrow \quad \beta[\boldsymbol{b}, \pi^{-1}(i)]\sigma_{\pi^{-1}(i)}(x) = \beta[\boldsymbol{b}', \pi^{-1}(i)]\sigma_{\pi^{-1}(i)}(x) \\
&\Leftrightarrow \quad \beta[\boldsymbol{b}, \pi^{-1}(i)]\sigma_{\pi^{-1}(i)}(y) = \beta[\boldsymbol{b}', \pi^{-1}(i)]\sigma_{\pi^{-1}(i)}(y) \\
&\Leftrightarrow \quad \phi(\beta)[\boldsymbol{c}, i](y) = \phi(\beta)[\boldsymbol{c}', i](y).
\end{aligned}$$

Finally, we define $\psi: \mathscr{C} \to \mathscr{B}$ by

$$\psi(\gamma)(b_0, \ldots, b_{n-1}) = \gamma(\sigma_{\pi^{-1}(0)}(b_{\pi^{-1}(0)}), \ldots, \sigma_{\pi^{-1}(n-1)}(b_{\pi^{-1}(n-1)})).$$

We have $\phi\psi = \mathrm{id}_{\mathscr{C}}$, and $\psi\phi = \mathrm{id}_{\mathscr{B}}$, so f is invertible. □

Theorem 2.9. *For all $n, k \geq 1$, we have*

$$|\mathrm{TC}_n(\underline{k})| = \sum_{k \geq k_0 \geq \cdots \geq k_{n-1} \geq 1} \begin{bmatrix} n \\ n_1, \ldots, n_m \end{bmatrix} \mathscr{S}_{k,k_0} \cdots \mathscr{S}_{k,k_{n-1}} |\mathrm{BL}_{k_0 \cdots k_{n-1}}(k)|,$$

where

$$\begin{bmatrix} n \\ n_1, \ldots, n_m \end{bmatrix} = \frac{n!}{n_1! \cdots n_m!}$$

is the number of permutations of the sequence $k_0, \ldots, k_{n-1}$, m being the number of distinct elements of that sequence and n_j ($j = 1, \ldots, m$) the number of times each value appears; the Stirling number of the second kind (*see* [1])

$$\mathscr{S}_{k,k_i} = \frac{1}{k_i!} \sum_{j=0}^{k_i} (-1)^j \begin{pmatrix} k_i \\ j \end{pmatrix} (k_i - j)^k$$

is the number of equivalence relations on $\underline{k}$ with exactly k_i blocks; and $\mathrm{BL}_{k_0 \cdots k_{n-1}}(k)$ is the set of all TC Latin bricks from $\prod_{i \in n} \underline{k}_i$ into $\underline{k}$.

Proof. We know, by Lemma 2.7, that $\mathrm{TC}_n(\underline{k})$ is partitioned by the equivalence $\beta = \{(f,g) \in \mathrm{TC}_n(\underline{k}) : \beta_f = \beta_g\}$, and that each block $[f]$ of β has cardinality $|\mathscr{R}_{B_0 \cdots B_{n-1}}| = |\{(\varrho_0, \ldots, \varrho_{n-1}) : \varrho_0, \ldots, \varrho_{n-1}$ are equivalences on $\underline{k}$ such that $B_i = \{\min[x]_{\varrho_i} : x \in \underline{k}\}$, $i = 0, \ldots, n-1\}|$, where $\beta_f : \prod_{i \in n} B_i \to \underline{k}$.

By Lemma 2.6, each TC Latin brick $\gamma : \prod_{i \in n} B_i \to \underline{k}$ with $0 \in B_i \subset k$ $\forall i$, defines a β-block, so we have

$$|\mathrm{TC}_n(k)| = \sum_{0 \in B_i \subset k\ (i \in n)} |\mathrm{BL}_{B_0 \cdots B_{n-1}}(k)|\,|\mathscr{R}_{B_0 \cdots B_{n-1}}|.$$

By Lemma 2.8, $|\mathrm{BL}_{B_0 \cdots B_{n-1}}(k)|$ depends only on the cardinalities of the B_i, hence

$$|\mathrm{TC}_n(\underline{k})| = \sum_{1 \leq k_0, \ldots, k_{n-1} \leq k} \; \sum_{\substack{0 \in B_i \subset \underline{k}, \\ |B_i| = k_i\ (i \in n)}} |\mathrm{BL}_{k_0 \cdots k_{n-1}}(k)|\,|\mathscr{R}_{B_0 \cdots B_{n-1}}|.$$

Now,

$$\sum_{\substack{0 \in B_i \subset k, \\ |B_i| = k_i\ (i \in n)}} |\mathscr{R}_{B_0 \cdots B_{n-1}}|$$

is precisely the number $\mathscr{S}_{k,k_0} \cdots \mathscr{S}_{k,k_{n-1}}$ of sequences of equivalences on k with respectively $k_0, \ldots, k_{n-1}$ blocks, which gives

$$|\mathrm{TC}_n(\underline{k})| = \sum_{1 \leq k_0, \ldots, k_{n-1} \leq k} \mathscr{S}_{k,k_0} \cdots \mathscr{S}_{k,k_{n-1}} |\mathrm{BL}_{k_0 \cdots k_{n-1}}(k)|.$$

Finally, since $|\mathrm{BL}_{k_0\cdots k_{n-1}}(k)|$ does not depend on the order of $k_0,\dots,k_{n-1}$, we can take the sum on the decreasing sequences $k_0,\dots,k_{n-1}$, multiplying by the appropriate factor, i.e. the number of possible permutations of those sequences. □

The last theorem allows us to compute $\mathrm{TC}_n(\underline{k})$, provided we know the number of TC Latin bricks defined on each product of n subsets $\underline{k}_0,\dots,\underline{k}_{n-1}$ of $\underline{k}$. This number is, in general, unknown; in [3], Gessel determined it for $n=2$ and $k_0=k$. In the next section, we will characterize those TC Latin bricks for which $n\geq 3$ and $k_0=k_1=k_2=k$. To finish this section, we illustrate the use of the formula of Theorem 2.9 by an example.

Example 2.10. We use Theorem 2.9 to compute $|\mathrm{TC}_2(\underline{4})|$. We have

$$|\mathrm{TC}_2(\underline{4})| = \sum_{4\geq k_0\geq k_1\geq 1} \begin{bmatrix} 2 \\ n_1,\dots,n_m \end{bmatrix} \mathcal{S}_{4,k_0}\mathcal{S}_{4,k_1}|\mathrm{BL}_{k_0k_1}(4)|.$$

In our case,

$$\begin{bmatrix} 2 \\ n_1,\dots,n_m \end{bmatrix}$$

is always equal to 1 or 2; also, we have $\mathcal{S}_{4,1}=1$, $\mathcal{S}_{4,2}=7$, $\mathcal{S}_{4,3}=6$ and $\mathcal{S}_{4,4}=1$. It remains to calculate $|\mathrm{BL}_{k_0k_1}(4)|$ for $(k_0,k_1)=(4,4)$, $(4,3)$, $(4,2)$, $(4,1)$, $(3,3)$, $(3,2)$, $(3,1)$, $(2,2)$, $(2,1)$ and $(1,1)$. These computations are too long to be given here so we give only the results:

$|\mathrm{BL}_{4,4}(4)|=|\mathrm{BL}_{4,3}(4)|=576$; $|\mathrm{BL}_{4,2}(4)|=216$; $|\mathrm{BL}_{4,1}(4)|=|\mathrm{BL}_{3,1}(4)|=24$; $|\mathrm{BL}_{3,3}(4)|=1\,056$; $|\mathrm{BL}_{3,2}(4)|=264$; $|\mathrm{BL}_{2,2}(4)|=84$; $|\mathrm{BL}_{2,1}(4)|=12$; and $|\mathrm{BL}_{1,1}(4)|=4$.

Finally, with all these we are able to obtain $|\mathrm{TC}_2(\underline{4})|=75\,328$.

We have compared our result to the lower and upper bounds found in [2] for $|\mathrm{TC}_n(\underline{k})|$; for $n=2$ and $k=4$, these bounds are respectively 16 384, and 140 737 488 355 328.

Berman recently wrote a computer program for calculating $|\mathrm{BL}_{k_0k_1}(k)|$ for any k, in the case $n=2$. The results for $k=5$ and $k=6$ are already very big (for instance, the number of 5×5 TC Latin bricks on six elements is more that $4\cdot 10^9$).

3. TC Latin Bricks induced by Abelian groups

We now turn our attention to TC Latin bricks $\beta:\prod_{i\in\underline{n}}\underline{k}_i\to\underline{k}$, where $n\geq 3$ and the first three k_i's are equal to k. It is well known that if we have a k by k Latin square on $\underline{k}$ that has one row equal to identity, it is not necessarily true that the k rows of the square form a permutation group. We will show that for TC Latin bricks in higher dimensions, this fact becomes true. We will need the following lemma.

Lemma 3.1. *Let G be a subgroup of S_k (the group of all permutations on $\underline{k}$) such that G has cardinality k, and $\sigma \neq \mathrm{id}_k \Rightarrow \sigma(0) \neq 0$ for all $\sigma \in G$. Then*

(i) *For all $\sigma, \tau \in G$ and $x \in \underline{k}$, we have $\sigma(x) = \tau(x) \Rightarrow \sigma = \tau$;*
(ii) *$G - \{\mathrm{id}_k\} \subset I_k$, where I_k is the set of fixed point free permutations on $\underline{k}$.*

Proof. We first prove (i) for $x=0$: if $\sigma \neq \tau$, then $\tau^{-1}\sigma \neq \mathrm{id}_k$, hence $\tau^{-1}\sigma(0) \neq 0$, i.e. $\sigma(0) \neq \tau(0)$. That means that the values $\sigma(0)$ ($\sigma \in G$) are all distinct, and since there are k of them, we have

$$\underline{k} = \{\sigma(0) \colon \sigma \in G\}.$$

To prove (ii), let $\sigma \in G$ and suppose $\sigma(x) = x$. There exists $\tau \in G$ with $\tau(0) = x$. We have

$$\tau(0) = x = \sigma(x) = \sigma\tau(0) \ \Rightarrow\ \tau = \sigma\tau \ \Rightarrow\ \sigma = \mathrm{id}_k.$$

Now (i) is a direct consequence of (ii), since $\sigma \neq \tau \Rightarrow \tau^{-1}\sigma \neq \mathrm{id}_k \Rightarrow \tau^{-1}\sigma(x) \neq x$ $\forall x \in \underline{k}$. □

Theorem 3.2. *Let β be a ternary operation on $\underline{k}$ such that $\beta[(0,0),0] = \mathrm{id}_k$ (see Definition 1.1). Then β is a TC Latin brick if and only if there is an Abelian subgroup $G = \{\mathrm{id}_k = \sigma_0, \sigma_1, \ldots, \sigma_{k-1}\}$ of S_k and a permutation $\pi \in S_k$, so that*

(1) $\sigma_i(0) \neq 0\ \forall i \geq 1$, *and* $\pi(0) = 0$;
(2) $\beta(x,y,z) = \sigma_{\pi(z)}\sigma_y(x)\ \forall (x,y,z) \in \underline{k}^3$.

Proof. ($\Rightarrow$) Suppose β is a TC Latin brick. Then its 0-derived functions $\beta[(y,0),0]$ are permutations of $\underline{k}$. Put $\sigma_y = \beta[(y,0),0]$ for $y = 1, \ldots, k-1$, and $\sigma_0 = \mathrm{id}_k = \beta[(0,0),0]$. We have for $y \neq y'$, $\beta(0,y,0) \neq \beta(0,y',0)$ because the 1-derived function $\beta[(0,0),1]$ is injective. So $\sigma_y(0) \neq \sigma_{y'}(0)$, which implies that σ_y and $\sigma_{y'}$ are distinct. Hence the σ_i's form a subset of S_k with cardinality k. Moreover, since $\sigma_0(0) = 0$, we have $\sigma_y(0) \neq 0$ for all $y \geq 1$.

Take $z > 0$. $\beta[(0,z),0]$ is a permutation of $\underline{k}$ as well; since $\{\sigma_y(0) \colon y \in \underline{k}\} = \underline{k}$, there exists $y \in \underline{k}$ such that $\beta[(0,z),0](0) = \sigma_y(0) = \beta[(y,0),0](0)$. By TC, this implies that $\beta[(0,z),0] = \beta[(y,0),0]$. Put $y = \pi(z)$, for each $z > 0$, and $0 = \pi(0)$. For $z \neq z'$, we have

$$\begin{aligned} \beta[(0,z),0] \neq \beta[(0,z'),0] &\Rightarrow \beta[(\pi(z),0),0] \neq \beta[(\pi(z'),0),0] \\ &\Rightarrow \pi(z) \neq \pi(z'). \end{aligned}$$

We have shown that π is a permutation of $\underline{k}$ that fixes 0, so (1) is completely proved. Next, take $x,y,z \in \underline{k}$. Since $\beta[(0,0),0] = \mathrm{id}_k$, we have $\beta(\beta(x,y,0),0,0) = \beta(x,y,0)$, so by TC, $\beta(\beta(x,y,0),0,z) = \beta(x,y,z)$. On the other hand, we have $\beta(x,y,0) = \beta[(y,0),0](x) = \sigma_y(x)$, and $\beta(\beta(x,y,0),0,z) = \beta[(0,z),0]\beta(x,y,0) = \beta(\pi(z),0),0]\beta(x,y,0) = \sigma_{\pi(z)}\beta(x,y,0) = \sigma_{\pi(z)}\sigma_y(x)$, so (2) is proved. Next, to prove that G is a subgroup of S_k, it is enough to show that it is closed under composition. The equation $\beta(x,y,z) = \sigma_{\pi(z)}\sigma_y(x)$ implies that the 0-derived functions of β are of the form $\sigma_{\pi(z)}\sigma_y$, with $y, z \in \underline{k}$; for $z=0$, these functions are equal to σ_y for some y. Since $\underline{k}=\{\sigma_y(0): y \in \underline{k}\}$, for each 0-derived function $\sigma_{\pi(z)}$ there exists a 0-derived function σ_b such that $\sigma_{\pi(z)}\sigma_y(0)=\sigma_b(0)$, and by TC, $\sigma_{\pi(z)}\sigma_y=\sigma_b$, i.e. $\sigma_{\pi(z)}\sigma_y \in G$. Now, π being a permutation of $\underline{k}$, the set of all $\sigma_{\pi(z)}\sigma_y$ ($y, z \in \underline{k}$) is GG, so G is closed under composition. It remains to see that it is Abelian. For all $x, y, z \in \underline{k}$, we have $\beta(x,0,z)=\beta(\beta(x,0,z),0,0)$, and by TC $\beta(x,y,z)=\beta(\beta(x,0,z),y,0)$. By (2), $\beta(x,y,z)=\sigma_{\pi(z)}\sigma_y(x)$, and $\beta(\beta(x,0,z),y,0)=\sigma_y\sigma_{\pi(z)}(x)$. Hence we have $\sigma_{\pi(z)}\sigma_y=\sigma_y\sigma_{\pi(z)}$.

($\Leftarrow$) Let G and π be as in (1) and (2). The 0-derived functions of β are of the form $\sigma_{\pi(z)}\sigma_y$, and since G is a group, they belong to G and are injective. Moreover, by Lemma 3.1, if $\sigma_{\pi(z)}\sigma_y(x)=\sigma_{\pi(z')}\sigma_{y'}(x)$ for some x, then $\sigma_{\pi(z)}\sigma_y=\sigma_{\pi(z')}\sigma_{y'}$. Hence two 0-derived functions are either equal, or different at each point.

The 1-derived functions are of the form $\beta[(a,c),1](y)=\sigma_{\pi(c)}\sigma_y(a)$. We have

$$\begin{aligned}\sigma_{\pi(c)}\sigma_y(a)=\sigma_{\pi(c)}\sigma_{y'}(a) \quad &\Leftrightarrow \quad \sigma_y(a)=\sigma_{y'}(a)\\ &\Leftrightarrow \quad y=y' \quad \text{by Lemma 3.1,}\end{aligned}$$

so $\beta[(a,c),1]$ is injective. If $\beta[(a',c'),1]$ is another 1-derived function, we have

$$\begin{aligned}&\sigma_{\pi(c)}\sigma_y(a)=\sigma_{\pi(c')}\sigma_y(a')\\ &\quad\Leftrightarrow \quad \sigma_y\sigma_{\pi(c)}(a)=\sigma_y\sigma_{\pi(c')}(a') \quad \text{by commutativity}\\ &\quad\Leftrightarrow \quad \sigma_{\pi(c)}(a)=\sigma_{\pi(c')}(a'),\end{aligned}$$

and this last equality does not depend on the variable y. The proof for the 2-derived functions is similar: β is a TC Latin brick. □

Corollary 3.3. *Let β be a ternary operation on $\underline{k}$ such that $\beta[(0,0),0]=\beta[(0,0),1]=\beta[(0,0),2]=\mathrm{id}_{\underline{k}}$. Then β is a TC Latin brick if and only if it is given by $\beta(x,y,z)=x+y+z$, where $(\underline{k};+,-,0)$ is an Abelian group.*

Proof. Suppose β is a TC Latin brick. Then there exists a group G and a permutation π satisfying (1) and (2) of Theorem 3.2. β is given by $\beta(x,y,z)=\sigma_{\pi(z)}\sigma_y(x)$. In particular, we have $y=\beta(0,y,0)=\sigma_y(0)$ and $z=\beta(0,0,z)=\sigma_{\pi(z)}(0)$ hence $z=\pi(z)$, i.e. π is identity. Put $x+y=\sigma_x(y)$ for all $x, y \in \underline{k}$.

(1) For all $\underline{k}$, we have $\sigma_x\sigma_y=\sigma_y\sigma_x$, and in particular $\sigma_x\sigma_y(0)=\sigma_y\sigma_x(0)$; by the remark we just made, this gives $\sigma_x(y)=\sigma_y(x)$, i.e. $+$ is commutative;

(2) $0+x=\sigma_0(x)=x$, so 0 is the neutral element;

(3) $(x+y)+z=\sigma_{x+y}(z)=\sigma_{\sigma_x(y)}(z)$. The index of $\sigma_{\sigma_x(y)}$ is given by the value this permutation takes at 0, i.e. $\sigma_x(y)=\sigma_x\sigma_y(0)$, hence $\sigma_{\sigma_x(y)}=\sigma_x\sigma_y$. We have $\sigma_{\sigma_x(y)}(z)=\sigma_x\sigma_y(z)=x+\sigma_y(z)=x+(y+z)$, i.e. $+$ is associative.

$(\underline{k};+,0)$ is an associative and Abelian quasigroup with neutral element 0; since $\underline{k}$ is a finite set, we can conclude that $(\underline{k};+,-,0)$ is an Abelian group. □

Theorem 3.2 extends to higher dimensions in the following manner.

Theorem 3.4. *Let $\beta: \coprod_{i\in \underline{n}} \underline{k}_i \to \underline{k}$ be a function with $n \geq 3$, $k = k_0 = k_1 = k_2 \geq k_3 \geq \cdots \geq 1$, and $\beta[\mathbf{0}, 0] = \mathrm{id}_{\underline{k}}$ (where $\mathbf{0}$ is the $(n-1)$-dimensional zero vector). Then β is a TC Latin brick if and only if there exists an Abelian subgroup $G = \{\mathrm{id}_k = \sigma_0, \sigma_1, \ldots, \sigma_{k-1}\}$ of S_k and injections $\pi_i: \underline{k}_i \to \underline{k}$ $(i = 2, \ldots, n-1)$, such that*

(1) $\sigma_i(0) \neq 0$ $\forall i \geq 1$ *and* $\pi_i(0) = 0$ *for all* $i \in \{2, \ldots, n-1\}$;
(2) $\beta(x_0, \ldots, x_{n-1}) = \sigma_{\pi_{n-1}(x_{n-1})} \cdots \sigma_{\pi_2(x_2)} \sigma_{x_1}(x_0)$ *for all* $(x_0, \ldots, x_{n-1}) \in \coprod_{i \in \underline{n}} \underline{k}_i$.

Proof. $(\Rightarrow)$ We induct on n. For $n = 3$, the assertion corresponds to the one in Theorem 3.2. Assume it is true for n, and let $\beta: \coprod_{i \in \underline{n+1}} \underline{k}_i \to \underline{k}$ be a TC Latin brick such that $k = k_0 = k_1 = k_2 \geq \cdots \geq k_n \geq 1$ and $\beta[\mathbf{0}, 0] = \mathrm{id}_{\underline{k}}$. Then $\beta[0, \underline{n}]$ is a TC Latin brick from $\prod_{i\in \underline{n}} \underline{k}_i$ into $\underline{k}$ and we have $\beta[0, \underline{n}][\mathbf{0}, 0] = \beta[\mathbf{0}, 0] = \mathrm{id}_{\underline{k}}$ (here $\mathbf{0}$ denotes any 0 vector in any dimension). By induction, there exists a group G and injections $\pi_2, \ldots, \pi_{n-1}$, satisfying (1) and (2) for $\beta[0, \underline{n}]$. Equation (2) becomes

$$\beta(x_0, \ldots, x_{n-1}, 0) = \sigma_{\pi_{n-1}(x_{n-1})} \cdots \sigma_{\pi_2(x_2)} \sigma_{x_1}(x_0).$$

We must define an injection π_n so that (1) and (2) are true for β. Now for all $x_n \in \underline{k}_n$, there is x_1 in $\underline{k}$ such that $\beta(0, \ldots, 0, x_n) = \sigma_{x_1}(0) = \beta(0, x_1, 0, \ldots, 0)$. By TC^{n-1}, we have

$$\beta(x_0, 0, x_2, \ldots, x_{n-1}, x_n) = \beta(x_0, x_1, x_2, \ldots, x_{n-1}, 0) \quad \text{for all } x_0, x_2, \ldots, x_{n-1}. \quad (*)$$

Put $\pi_n(x_n) = x_1$. We have $\pi_n(0) = 0$, and $\pi_n(x_n) \neq \pi_n(x_n')$ if $x_n \neq x_n'$, since $\beta(0, \pi_n(x_n), 0, \ldots, 0) \neq \beta(0, \pi_n(x_n'), 0, \ldots, 0)$. Take $(x_0, \ldots, x_n) \in \prod_{i \in \underline{n+1}} \underline{k}_i$. By $(*)$ we have

$$\begin{aligned}\beta(x_0, 0, x_2, \ldots, x_{n-1}, x_n) &= \beta(x_0, \pi_n(x_n), x_2, \ldots, x_{n-1}, 0)\\ &= \sigma_{\pi_{n-1}(x_{n-1})} \cdots \sigma_{\pi_2(x_2)} \sigma_{\pi_n(x_n)}(x_0)\\ &= \sigma_{\pi_{n-1}(x_{n-1})} \cdots \sigma_{\pi_2(x_2)} \sigma_{\pi_1(0)}(\sigma_{\pi_n(x_n)}(x_0))\\ &= \beta(\sigma_{\pi_n(x_n)}(x_0), 0, x_2, \ldots, x_{n-1}, 0),\end{aligned}$$

and hence by TC,

$$\begin{aligned}\beta(x_0, x_1, x_2, \ldots, x_{n-1}, x_n) &= \beta(\sigma_{\pi_n(x_n)}(x_0), x_1, x_2, \ldots, x_{n-1}, 0)\\ &= \sigma_{\pi_{n-1}(x_{n-1})} \cdots \sigma_{\pi_2(x_2)} \sigma_{x_1} \sigma_{\pi_n(x_n)}(x_0),\end{aligned}$$

which by commutativity yields (2).

$(\Leftarrow)$ The proof is without induction and similar to the one in Theorem 3.2. $\square$

Corollary 3.5. *Let $\beta: \prod_{i \in \underline{n}} \underline{k}_i \to \underline{k}$, where $n \geq 3$ and $k = k_0 = k_1 = k_2 \geq k_3 \geq \cdots \geq k_{n-1} \geq 1$, be any function. Then β is a TC Latin brick if and only if there exists an Abelian subgroup $G = \{\mathrm{id}_k = \sigma_0, \sigma_1, \ldots, \sigma_{k-1}\}$ of S_k, injections $\pi_i: \underline{k}_i \to \underline{k}$ $(i = 2, \ldots, n-1)$ and a permutation $\tau \in S_k$, such that*

(1) $\sigma_i(0) \neq 0$ $\forall i \in \{1, \ldots, k-1\}$ *and* $\pi_i(0) = 0$ $\forall i \in \{2, \ldots, n-1\}$;
(2) $\beta(x_0, \ldots, x_{n-1}) = \tau\sigma_{\pi_{n-1}(x_{n-1})} \cdots \sigma_{\pi_2(x_2)} \sigma_{x_1}(x_0)$ $\forall (x_0, \ldots, x_{n-1}) \in \coprod_{i \in \underline{n}} \underline{k}_i$.

Proof. ($\Rightarrow$) Assume β is a TC Latin brick. Then $\tau = \beta[\mathbf{0}, 0]$ is a permutation of $\underline{k}$. Set $\gamma = \tau^{-1}\beta$. It is easy to see that γ is a function from $\prod_{i \in \underline{n}} \underline{k}_i$ into $\underline{k}$ which is a TC Latin brick if and only if β is. Now apply Theorem 3.4 to γ to obtain (1) and (2).

($\Leftarrow$) Again put $\gamma = \tau^{-1}\beta$. By Theorem 3.4, γ is a TC Latin brick, and hence β is a TC Latin brick. $\square$

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